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MOTION OF GAS IN HALF-OPEN PIPES

The motion of a gas contained in a tube of constant cross section of length L which is closed at one end (x=0) and communicates with the air at the other end (x=L) is regulated by the equations of hydrodynamics

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \qquad (2)$$

where u is the component of the gas velocity, ρ its density, p its pressure, all of which are functions of x and the time t. In these equations density and pressure are related by the adiabatic equation of state

$$p = f(\rho) = k\rho^{\gamma} \qquad (3)$$

The boundary conditions are

$$u = 0 \text{ at } x = 0 \qquad (4)$$

and

$$\rho = \rho_0 \text{ , } p = p_0 \text{ at } x = L \qquad (5)$$

together with the prescription of the variables u and ρ along the tube for initial time t = 0. The boundary condition at the open end is subject to discussion, and experiments in acoustics have shown that even in the case of infinitesimal vibrations of an organ pipe the above boundary condition at the end of the pipe should be replaced by

$$p = p_0 \quad \text{at } x = L'$$

where L' is the so-called effective length and differs slightly from L.

The problem as formulated thus far is not always capable of a solution for all time because of the well known occurence of pressure discontinuities which may appear later although the initial state is perfectly regular. But even if we disregard this difficulty the numerical solution of the above partial differential equations is a matter of considerable complexity and the study by numerical integration of the dependence of the motion on the initial data seems to involve an almost prohibitive amount of labor. Hence the question arises whether the above flow problem can be replaced by a simpler one in which the details of the velocity and pressure distribution along the length of the tube are sacrificed for the sake of simplicity and the state of the gas is described by only two variables, the pressure $\bar{p}$ (or density $\bar{\rho}$) at the closed end and the velocity U at the open end. Such substitution is inadequate to portray the multiplicity of phenomena which the varieties of initial states may produce in the exit velocities and pressures at the closed end as time progresses. We believe, however, that from the solution of the simplified problem below one can gain information of qualitative nature about the effects if a change of initial pressure and shape of the tube on the duration and magnitude of the thrust on the tube.

Straight Pipe. In order to derive the simplified system mentioned let us integrate (1) and (2) with respect to x from 0 to L. We obtain

$$(6) \qquad \frac{\partial}{\partial t}\int_0^L u\,dx + \frac{1}{2}U^2 = \frac{\gamma k}{\gamma-1}\left[\bar{\rho}^{\gamma-1} - \rho_0^{\gamma-1}\right]$$

(7) $$\frac{\partial}{\partial t}\int_0^L(\rho - \rho_o)dx = - \rho_o U$$

We write

(8) $$\int_0^L u dx = k_1 U,$$

(9) $$\int_0^L(\rho - \rho_o)dx = k_2(\bar{\rho} - \rho_o)$$

where k_1 and k_2 are functions of t. The simplifying assumption we shall make is to neglect this dependence of $k_1 k_2$ on t, i.e. to take for k_1 and k_2 constant values to be determined later on. With this substitution we obtain from (6), (7) the following two equations

(10) $$k_1\frac{dU}{dt} + \frac{1}{2}U^2 = \frac{1}{\gamma - 1}c_o^2\left[\left(\frac{\bar{\rho}}{\rho_o}\right)^{\gamma-1} - 1\right] \text{ (where } c_o^2 = \gamma\,\rho_o^{\gamma-1})$$

(11) $$k_2\frac{d}{dt}\frac{\bar{\rho}}{\rho_o} = - U$$

which reduce to the one equation of second order

(12) $$k_1k_2\frac{d^2\sigma}{dt^2} - \frac{1}{2}k_2^2\left[\frac{d\sigma}{dt}\right]^2 = \frac{c_o^2}{\gamma - 1}(1 - \sigma^{\gamma-1}),$$ (13) $$\sigma = \frac{\bar{\rho}}{\rho_o}.$$

We are concerned with the motion produced by an initial excess pressure near the closed end and initial velocity zero; here the following procedure suggests itself for the determination of k_1 and k_2. The simplest such motion for which the equations (8), (9) hold exactly with constant k_1 and k_2 is the first mode

of the infinitesimal vibrations of the gas in the tube. Hence we shall use this first mode for the determination of k_1 and k_2. Since we have in this case

$$u = U \sin(\frac{\pi}{2}\frac{x}{L}), \quad \rho - \rho_o = (\bar{\rho} - \rho_o)\cos(\frac{\pi}{2}\frac{x}{L})$$

we find from (8) and (9)

(14) $$k_1 = \frac{2L}{\pi}, \quad k_2 = \frac{2L}{\pi}.$$

Thus (12) becomes

(15) $$\frac{d^2\sigma}{dt^2} - \frac{1}{2}\left(\frac{d\sigma}{dt}\right)^2 = \frac{c_o^2}{\gamma - 1}\frac{\pi^2}{4L^2}(1 - \sigma^{\gamma-1})$$

and upon introducing the dimensionless variable.

(16) $$\tau = \frac{c_o\pi}{2L}t$$

we find

(17) $$\frac{d^2\sigma}{d\tau^2} - \frac{1}{2}\left(\frac{d\sigma}{d\tau}\right)^2 = \frac{1}{\gamma - 1}(1 - \sigma^{\gamma-1}).$$

The variable τ is normed so that 2π units of τ correspond to a full cycle of the infinitesimal vibration, i.e., to a time T given by

$$T = \frac{4L}{c_o}.$$

Equation (17) can be solved by quadratures. Left hand is

$$e^{\frac{\sigma}{2}}\frac{d}{d\tau}\left(e^{-\sigma/2}\frac{d\sigma}{d\tau}\right) = e^{\frac{\sigma}{2}}\frac{d\sigma}{d\tau}\frac{d}{d\sigma}\left(e^{-\frac{\sigma}{2}}\frac{d\sigma}{d\tau}\right)$$

$$= \frac{1}{2}e^{\sigma}\frac{d}{d\sigma}\left(e^{-\frac{\sigma}{2}}\frac{d\sigma}{d\tau}\right)^2$$

Hence

(18) $$\frac{d}{d\sigma}\left(e^{-\frac{\sigma}{2}}\frac{d\sigma}{d\tau}\right)^2 = \frac{2\,e^{-\sigma}}{\gamma-1}\left(1-\sigma^{\gamma-1}\right),$$

(19) $$e^{-\frac{\sigma}{2}}\frac{d\sigma}{d\tau} = \sqrt{\frac{2}{\gamma-1}\int_{\sigma_i}^{\sigma} e^{-\sigma}\left(1-\sigma^{\gamma-1}\right)d\sigma}$$

where σ_1 is an initial value (> 1), and $\frac{d\sigma}{d\tau}$ is taken as zero initially because of the requirement of zero initial velocity and (11). From (19) τ follows as function of σ by a quadrature.

Discussion of Numerical Results for Straight Pipe. The numerical results are seen in the black σ, τ curves of graph 1; they are drawn to cross the line $\sigma = 1$ for $\tau = 0$. γ is set equal to 1.2. The ordinates give the ratio σ of densities at closed and open ends of the pipe, for initial σ_1 = 2.1, 3.3, 4.5, 7.0. Alongside the density ratios σ are given the corresponding pressure ratios; their initial values are accordingly 2.4, 4.2, 6.1, 10.3.

The resulting curves show the remarkable feature that the time τ_0 elapsed between $\sigma = \sigma_1$ and $\sigma = 1$ is close to linear in σ_1 of the form

(20) $$\tau_0 = \frac{\pi}{2}\left(1 + k(\sigma_1 - 1)\right)$$

with $k = .258$; similarly the time τ_1 from $\sigma = 1$ to $\sigma = \sigma_{min}$ is given by

(21) $$\tau_1 = \frac{\pi}{2}\left(1 - k(1 - \sigma_{min}\right).$$

Since the value of σ_{m1} corresponding to a given σ_1 varie but little with σ_1, only the two curves with $\sigma_1 = 7.0$ and 2.1 are continued below $\sigma = 1$; clearly the "suction" time τ_1 is diminished but little as the initial excess pressure increases.

The point of inflection $\frac{d^2\sigma}{d\tau^2} = 0$, always occurs for a value of τ about midway between the time of maximum and minimum pressures; this point corresponds to maximum exit velocity U, according to (11); it is reached before the pressure has declined to p_0. Fig. 3 indicates the variation of the "total positive $\int(p - 1)dt$ thrust" during the time of decline of the pressure from its maximum p_1 to p_0, shown as function of the initial relative excess pressure $\frac{p_1 - p_0}{p_0}$.

<u>Pipes of Varying Cross-Section</u>. The foregoing treatment can be modified so as to allow for the effect of a slowly varying cross-section A(x) on the motion of the gas. Equation (2) is replaced by

$$(2') \qquad A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(A\rho u) = 0 \quad .$$

Accordingly (7) becomes

$$(7') \qquad \frac{d}{dt}\int_0^L A(\rho - \rho_0)dx = - A(L)\,\rho_0 U.$$

(8) remains as definition of k_1; (9) is replaced by

$$(9') \qquad \int_0^L A(\rho - \rho_0)dx = k_2(\bar{\rho} - \rho_0)A(L)$$

and again the simplifying assumption is made that k_1 and k_2 remain constant. Then we find again (10) and (11) from which (12) follows

for σ defined by (13). Upon introducing the new time

$$\tau = \frac{c_o t}{\sqrt{k_1 k_2}} \tag{16'}$$

we have from (12)

$$\frac{d^2\sigma}{d\tau^2} - \frac{k_2}{2k_1}\left(\frac{d\sigma}{d\tau}\right)^2 = \frac{1}{\gamma - 1}\left(1 - \sigma^{\gamma - 1}\right) \tag{17'}$$

which differs from (17) only in that $\frac{k_2}{k_1}$ is not necessarily equal to unity.

The treatment of the simplified motion now consists of two independent parts: a) the determination of k_1 and k_2; b) the integration of (17').

Determination of k_1 and k_2 from the Infinitesimal Oscillation. As in the case of a straight pipe we are led to attempt the calculation of k_1 and k_2 with the aid of the theory of small vibrations of the pipe. Then we set

$$u = g(t)\,\frac{df(x)}{dx}$$

with $f(x)$ and $g(t)$ to be determined, $\frac{df}{dx} = 0$ at $x = 0$. From (1) there follows, after familiar pattern, that

$$\frac{\partial\left(\frac{\rho - \rho_o}{\rho_o}\right)}{\partial x} = -\frac{1}{c_o^2}\,\frac{\partial u}{\partial t}\text{, hence}$$

$$\frac{\rho - \rho_o}{\rho_o} = -\frac{g(t)}{c_o^2}\,f(x);\; f(L) = 0.$$

(2') yields

$$-Af\frac{\ddot{g}}{c_0^2} + \frac{d}{dx}\left(A\,\frac{df}{dx}\right)g(t) = 0$$

whence

$$\frac{d}{dx}\left(A\,\frac{df}{dx}\right) + \lambda^2 Af = 0 \tag{22}$$

and

$$\ddot{g} + \lambda^2 c_0^2\, g = 0, \quad \text{or } g = \begin{matrix}\sin\\ \cos\end{matrix}\ \lambda c_0 t.$$

Thus the period of the infinitesimal vibration becomes

$$T = \frac{2\pi}{\lambda c_0}$$

while the definitions (8) and (9') give

$$k_1 U = k_1 g(t)\,\frac{df}{dx}(L) = \int_0^L u\,dx = g(t)\,(f(L) - f(0))$$

$$A(L)k_2(\bar{\rho} - \rho_0) = -A(L)k_2\,\frac{\rho_0}{c_0^2}\,g(t)f(0) = \int_0^L A(\rho - \rho_0)\,dx$$

$$= -\frac{g(t)}{c_0^2}\,\rho_0\int_0^L Af\,dx = +\frac{g(t)}{c_0^2}\,\frac{\rho_0}{\lambda^2}\int_0^L \frac{d}{dx}\left(A\,\frac{df}{dx}\right)dx$$

$$= \frac{g(t)}{c_0^2}\,\frac{\rho_0}{\lambda^2}\left[\left(A\,\frac{df}{dx}\right)\right]_0^L .$$

The boundary conditions (23) $f(L) = 0$, $\frac{df}{dx}(0) = 0$ give

$$k_1 = -\frac{f(0)}{\frac{df}{dx}(L)}, \quad k_2 = -\frac{\frac{df}{dx}(L)}{\lambda^2 f(0)} . \tag{24}$$

Hence

$\lambda^2 = \frac{1}{k_1 k_2}$, so that the period of the first mode becomes

$$T = \frac{2\pi\sqrt{k_1 k_2}}{c_0} = \frac{2\pi}{\lambda c_0^{\xi}}.$$

This shows that one cycle corresponds to 2π units of τ as defined by (16').

The value of λ is the minimum number for which (22) has a non-vanishing solution under the boundary conditions (23).

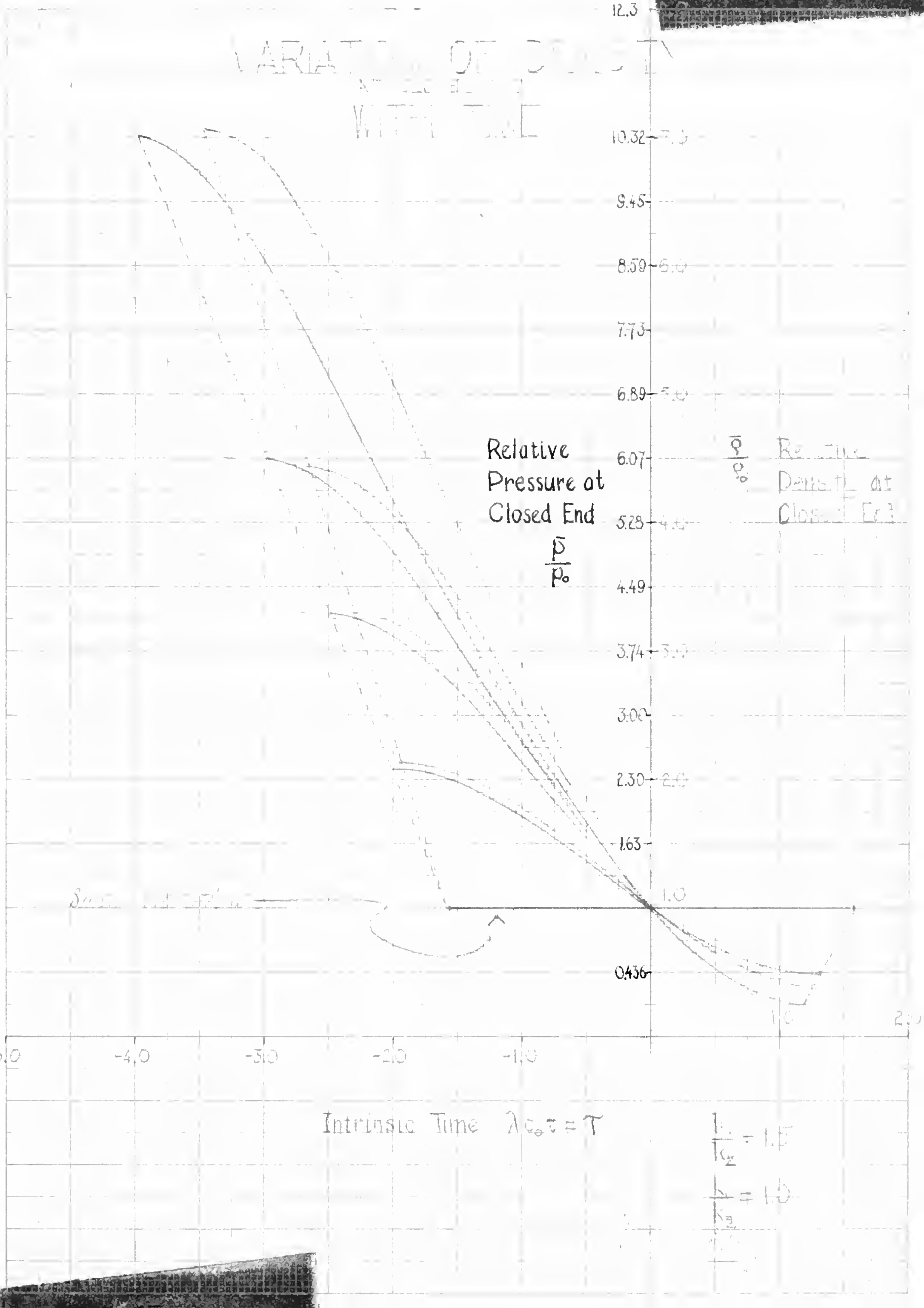
Consider two pipes I and II for which the cross sections are A_1 and A_2, the λ values λ_1 and λ_2. It can be shown that $\lambda_1 < \lambda_2$ if the ratio $\frac{A_2(x)}{A_1(x)}$ increases somewhere as x goes from 0 to L without ever decreasing in this interval. Thus flaring the pipe toward the open end will decrease the period T, while contraction toward the open end will increase the period compared with that of the straight pipe.

<u>Shapes Considered Numerically.</u> We have computed λ, k_1, k_2 for the shapes shown in the accompanying figures. Notice that only the relative values of the cross section areas are relevant to the motion considered.

<u>Integration of the Differential Equation.</u> (17'). The results of integration, which can be carried in a manner analogous to that of (17), are shown on graph (1) in the blue and red curves corresponding

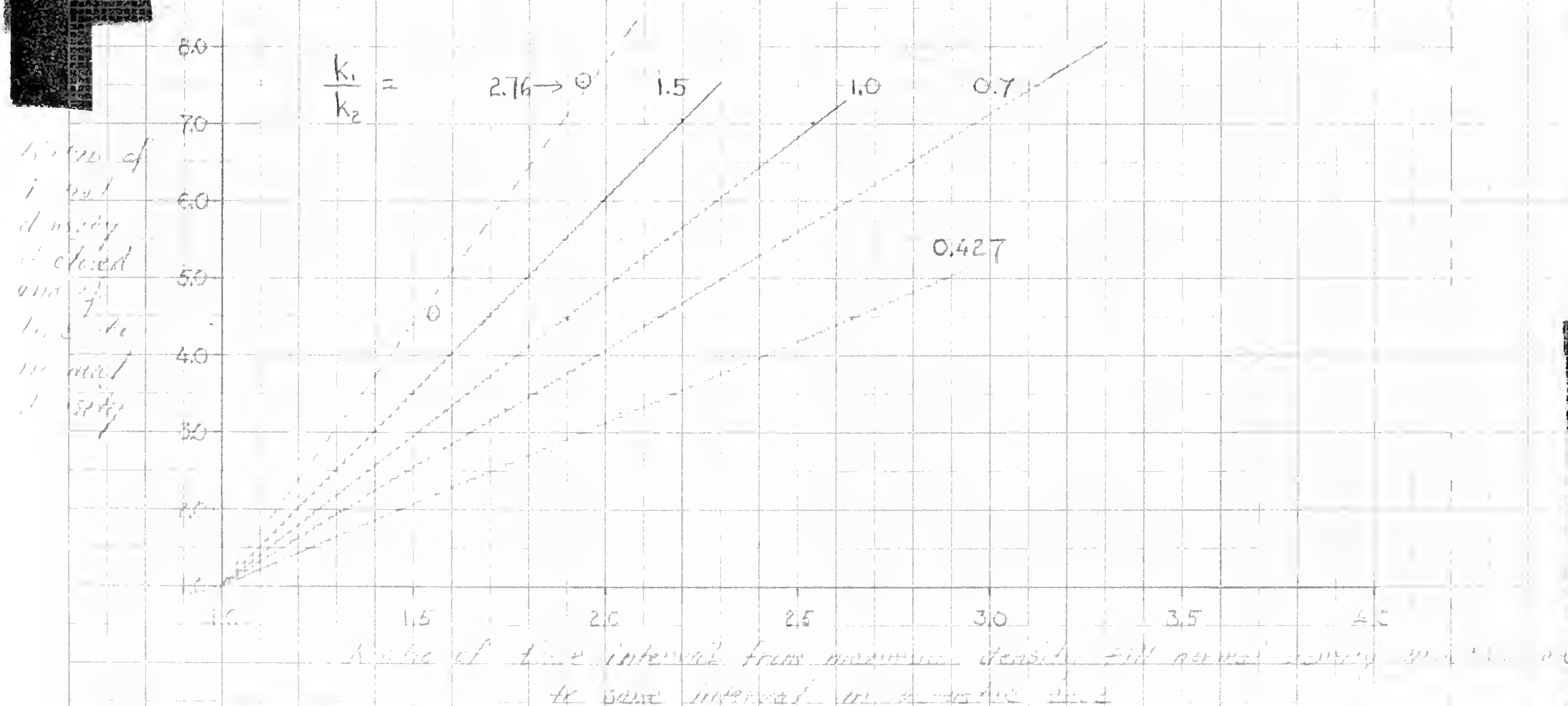
to $\frac{k_1}{k_2}$ = 1.5 and .7 respectively. Notice that when $\frac{k_1}{k_2}$ <u>decreases</u> the time of excess-pressure increases if the unit of time is chosen as $\frac{1}{2\pi}$ of the period of the first acoustic mode; this, incidentally, is the unit in which our $\mathcal{T}$ is measured. Figure 2 is a plot of the excess-pressure time in $\mathcal{T}$ units versus σ_1 i.e. versus the ratio of the initial density at the closed end to the density at the open end.

It can be proved that flaring the pipe increases $\frac{k_1}{k_2}$ as well as λ . Thus, flaring the pipe not only produces , for identical initial pressures, a shorter time interval of excess pressure <u>when measured in multiples of the fundamental acoustic period</u>, but this acoustic period, too, is shortened by the flaring of the pipe. The reverse holds for contracting the pipe toward the open end.

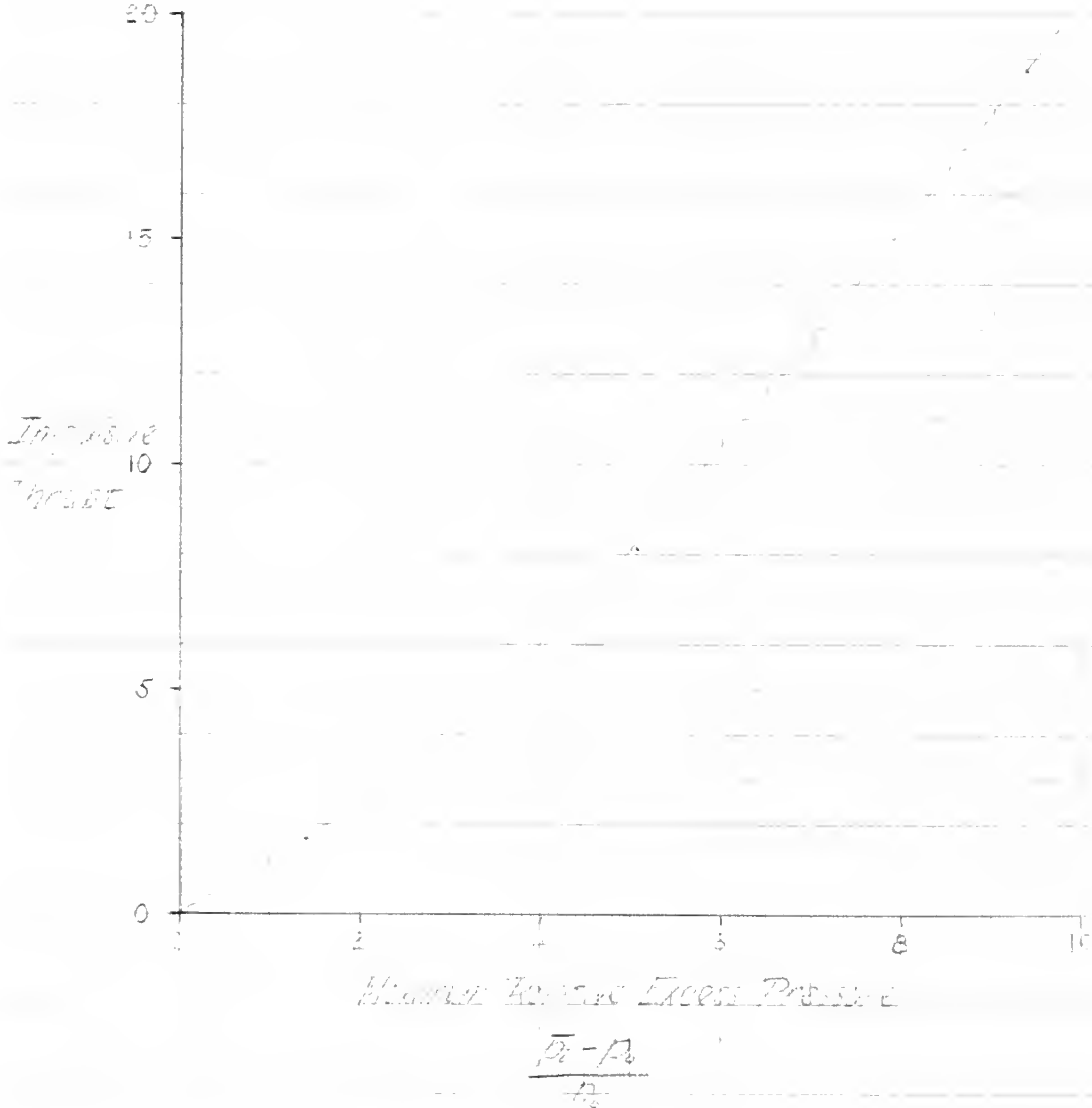


WITH TIME
Relative
Pressure at
Closed End
p̄/p₀
12.3
10.32
9.45
8.59
7.73
6.89
6.07
5.28
4.49
3.74
2.30
1.63
1.0
0.436
Intrinsic Time λc₀t = T
-4.0
-3.0
-2.0
-1.0
1.0
2.0

VARIATION OF INTERVAL FROM MAXIMUM TILL NORMAL DENSITY WITH MAGNITUDE OF MAXIMUM



VARIATION OF POSITIVE IMPULSIVE THRUST WITH INITIAL PRESSURE



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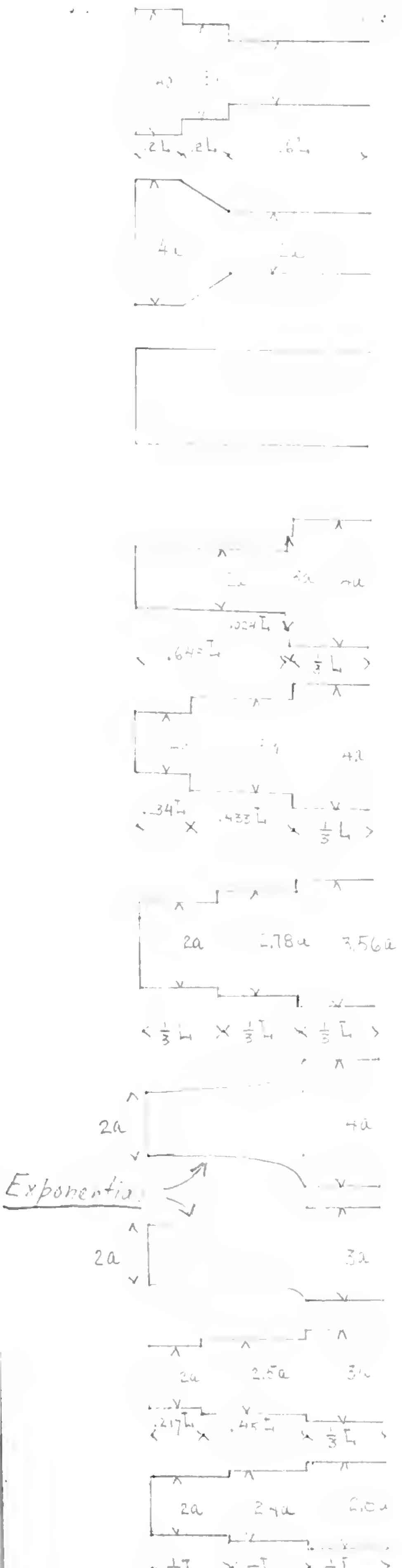

λL	$\frac{k_1}{k_2}$
1.027	.427
1.015	.430
1.57	1.
2.083	9.5
2.083	2.8
2.083	2.9
2.083	2.0
1.868	1.6
1.868	1.89
1.868	1.90

Fig. 4

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Motion of gas in half-open pipes

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Motion of gas in half-open pipes.